ON OPTIMAL CONTROL OF PROCESSES IN DISTRIBUTED OBJECTS

(OB OPTIMAL'NOM UPRAVLENII PROTSESSAMI V RASPREDELENNYKH OB'EKTAKH)

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A number of practical problems have let to a need for studying controlled processes in systems with distributed parameters. Butkovskii and Lerner [1] formulated in its most general form the problem of optimal control of processes in such systems and showed that in certain cases this problem may be solved by applying Pontriagin's maximum principle [2]. Later Butkovskii [3,4] found the optimum conditions for the case in which the controlled process is described by the nonlinear integral equation

$$Q(P) = \int_D K(P, S, Q(S), u(S))\, dS$$

where $Q(P)$ is a vector function characterizing the state of the controlled system, m is a vector function which is nonlinear in Q and u in the general case, $u(S)$ is an admissible control function, and D is a region in m-dimensional Euclidean space.

In the present note we formulate one type of problem concerning the optimal control of processes which are described by systems of quasilinear partial differential equations. Rozonoer's method [5] is used in obtaining the necessary optimum conditions. The article also indicates the optimum conditions, sufficient in the local sense, when the process is described by a system of linear equations.

Let the controlled process be described by the equations

$$L_i[u_i] \equiv \frac{\partial^2 u_i}{\partial x \partial t} + b_i(x, t)\frac{\partial u_i}{\partial x} + c_i(x, t)\frac{\partial u_i}{\partial t} = f_i(x, t, u_1, \ldots, u_n, v) \quad (i = 1, \ldots, n) \qquad (1)$$

where the functions b_i and c_i have continuous second derivatives with respect to x and t in the region G $(0 \leqslant x \leqslant l,\ 0 \leqslant t \leqslant T)$, v is a controlling parameter which assumes values in the (open or closed) convex domain V of some r-dimensional Euclidean space. The functions f_i are assumed to be continuous in x and t and twice continuously differentiable with respect to $u_1, \ldots, u_n$ and v.

We shall assume that the functions u_i satisfy the additional conditions (Goursat conditions)

$$u_i(x, 0) = \varphi_i(x), \qquad u_i(0, t) = \psi_i(t) \tag{2}$$

where the functions φ_i and ψ_i are continuously differentiable and satisfy the matching conditions $\varphi_i(0) = \psi_i(0)$.

For our class of admissible control functions we shall take the set of bounded functions piecewise continuous in x and t and defined in the domain G with values in V.

If the control function $v(x, t)$ has a discontinuity on a line parallel to one of the coordinate axes, for example on the line $t = a$

$$v(x, t) = \begin{cases} v_1(x, t) & \text{if } 0 \leqslant t < a,\ 0 \leqslant x \leqslant l \\ v_2(x, t) & \text{if } a < t \leqslant T,\ 0 \leqslant x \leqslant l \end{cases}$$

where $v_i(x, t)$ are continuous functions, then the corresponding solution of the problem (1) to (2) in the region G may be constructed sequentially. We first solve the problem

$$L_i[u_i] = f(x, t, u_1, \ldots, u_n, v_1(x, t)), \qquad u_i(x, 0) = \varphi_i(x), \qquad u_i(0, t) = \psi_i(t)$$
$$(0 \leqslant t \leqslant a,\ 0 \leqslant x \leqslant l)$$

This has a unique solution which is twice continuously differentiable with respect to x and t: $u_i = u_i^1(x, t)$ [6, pp. 63-67]. In like manner we find the solution $u_i = u_i^2(x, t)$ of the problem

$$L_i[u_i] = f(x, t, u_1, \ldots, u_n, v_2(x, t)), \qquad u_i(x, a) = u_i^1(x, a), \qquad u_i(0, t) = \psi_i(t)$$
$$(a \leqslant t \leqslant T,\ 0 \leqslant x \leqslant l)$$

Consequently, we have a continuous solution

$$u(x, t) = \begin{cases} u^1(x, t) & \text{if } 0 \leqslant t < a \\ u^2(x, t) & \text{if } a \leqslant t \leqslant T \end{cases}$$

corresponding to the discontinuous control function.

However, on the line of discontinuity $t = a$ the functions $u_i(x, t)$ do not have continuous derivatives. If the function $v(x, t)$ is regular [7, p. 19], then corresponding to this function we have a unique solution $u(x, t) = \{u_1, \ldots, u_n\}$, such that the functions $u_i(x, t)$ have continuous first derivatives with respect to x and t and integrable second derivatives. In this case the solution of the problem (1) to (2) is not unique within the class of continuous functions (see Example 2).

For this reason we may always assume that to each admissible control function $v(x, t)$ there corresponds a unique solution $u(x, t)$ which has continuous derivatives with respect to x and t if $v(x, t)$ is regular and is continuous if $v(x, t)$ is not regular.

Let A_i $(i = 1, \ldots, n)$ be a given system of real numbers; let $\alpha_i(x)$, $\beta_i(t)$ and $\gamma_i(x, t)$ be given functions continuous in the region G. We take the admissible control function $v(x, t)$, denote by $u(x, t)$ the solution of the problem (1) to (2) corresponding to this control function, and consider the functional

$$S = \sum_{i=1}^{n} \Big[A_i u_i (l, T) + \int_0^l \alpha_i (x)\, u_i (x, T)\, dx + \int_0^T \beta_i (t)\, u_i (l, t)\, dt + \int_0^l \int_0^T \gamma_i (x, t)\, u_i (x, t)\, dt\, dx\Big]$$

where l and T are constants appearing in the definition of the region G.

Among all the admissible control functions, it is required to find a control function $v(x, t)$ such that the functional S attains its minimum (maximum) value.

The admissible control function for which a minimum (maximum) of this functional is attained will be called min-optimal (max-optimal) with respect to S.

It should be noted, by the way, that a boundary value problem of this kind is of great interest from the viewpoint of physical applications. It is encountered in the study of gas sorption and desorption processes [8], drying processes [9], etc. The presence of the parameter v in equations (1) makes it possible to control the process and in many cases to select the best operation, which (from the mathematical viewpoint) is equivalent to minimizing or maximizing some functional. In a number of cases the problem is reducible to a study of the functional S.

For example, let the controlled process be described by equations (1) with conditions (2) and let it be required to minimize the functional

$$J = \int_0^l \int_0^T f_0 (x, t, u, v)\, dx\, dt$$

We introduce the auxiliary variable u_0 by means of the equation

$$\frac{\partial^2 u_0}{\partial x\,\partial t} = f_0(x, t, u, v) \tag{3}$$

and the supplementary conditions

$$u(0, t) = u(x, 0) = 0 \tag{4}$$

Then the problem reduces to minimizing the functional $S = u_0(l, T)$, defined on the functions $u_0, \ldots, u_n$ given by equations (1) and (3) and the supplementary conditions (2) and (4). To obtain a solution, we introduce the auxiliary functions $w_i(x, t)$ by means of the equations

$$M_i[w_i] \equiv \frac{\partial^2 w_i}{\partial x\,\partial t} - \frac{\partial}{\partial x}(b_i w_i) - \frac{\partial}{\partial t}(c_i w_i) = \sum_{\nu=1}^{n} \frac{\partial f_\nu(x, t, u, v)}{\partial u_i} w_\nu - \gamma_i(x, t) \tag{5}$$

with the supplementary conditions

$$\begin{aligned} w_i(x, T) &= \int_l^x \alpha_i(\xi) \exp\left(\int_\xi^x c_i(\xi, T)\, d\xi\right) d\xi - A_i \exp\left(\int_l^x c_i(\xi, T)\, d\xi\right) \\ w_i(l, t) &= \int_T^t \beta_i(\tau) \exp\left(\int_\tau^t b_i(l, \tau)\, d\tau\right) d\tau - A_i \exp\left(\int_T^t b_i(l, \tau)\, d\tau\right), \quad (i = 1, \ldots, n) \end{aligned} \tag{6}$$

where the constants A_i and the functions $\alpha_i(x)$, $\beta_i(t)$ and $\gamma_i(x, t)$ are taken from the functional S. The system of equations (5) is linear, and so are the conditions (6). Therefore, to each admissible control function there corresponds a unique vector function $w(x, t) = \{w_1, \ldots, w_n\}$ which is defined in the region G and satisfies the system of equations (5) and the supplementary conditions (6). We set

$$H(u(x, t), w(x, t), v, x, t) = \sum_{i=1}^{n} w_i(x, t) f_i(x, t, u(x, t), v)$$

We shall say that the control function $v(x, t)$ satisfies a maximum condition if

$$H(u(x, t), w(x, t), v(x, t), x, t) \;((=))\; \sup H(u(x, t), w(x, t), v, x, t)(v \in V) (\mathrm{H}),$$

where the symbol $((=))$ represents equality that is valid at all points of the region G except possibly for a set of points that lie on a finite number of lines and the zero plane.

The minimum condition is determined in a similar manner.

Theorem 1 (Maximum principle). In order that an admissible control function $v(x, t)$ be min-optimal (max-optimal) with respect to S, it must satisfy a maximum (minimum) condition.

To prove this, we consider the functional

$$J[u, w, v] = \iint_G \Big[\sum_{i=1}^{n} w_i L_i[u_i] - H(u, w, v, x, t)\Big]\, dx\, dt$$

If $u = u(x, t)$ is the solution of the problem (1) to (2) corresponding to the control function v, then the functional is equal to zero for any arbitrary function $w = w(x, t)$.

Let an admissible control function $v(x, t)$ and the solution $u(x, t)$ corresponding to it be min-optimal with respect to S. We shall denote by $w(x, t)$ the solution of the problem (5) to (6) corresponding to the functions $v(x, t)$ and $u(x, t)$.

We shall denote by $u(x, t) + \Delta u(x, t)$ and $w(x, t) + \Delta w(x, t)$ functions which are solutions of the same problems but correspond to the admissible function $v(x, t) + \Delta v$, where $\Delta v(x, t)$ is some increment of the control function $v(x, t)$. Evidently the increments Δu_i and Δw_i will satisfy the equations

$$L_i[\Delta u_i] = \Delta \frac{\partial H}{\partial w_i}, \qquad M_i[\Delta w_i] = \Delta \frac{\partial H}{\partial u_i} \qquad (i = 1, \ldots, n) \tag{7}$$

and the boundary conditions

$$\Delta u_i(x, 0) = \Delta u_i(0, t) = \Delta w_i(x, T) = \Delta w_i(l, t) = 0 \tag{8}$$

where

$$\Delta \frac{\partial H}{\partial z_i} = \frac{\partial H(z + \Delta z, v + \Delta v, x, t)}{\partial z_i} - \frac{\partial H(z, v, x, t)}{\partial z_i}, \quad z = \{u_1, \ldots, w_n\} \tag{9}$$

We shall denote by ΔJ an increment of the functional J. Then, since the operators L_i are linear, we find

$$\Delta J = J[z + \Delta z, v + \Delta v] - J[z, v] = \iint_G \sum_{i=1}^{n} [\Delta w_i L_i[u_i] + \Delta w_i L_i[\Delta u_i] +$$
$$+ w_i L_i[\Delta u_i]]\, dx dt - \iint_G [H(z + \Delta z, v + \Delta v, x, t) - H(z, v, x, t)]\, dx\, dt = 0 \tag{10}$$

It is known [10, p.196] that for arbitrary functions $p(x, t)$ and $q(x, t)$ which are twice piecewise continuously differentiable with respect to x and t, the following equation (Green's theorem)

$$\iint_G [pL_i[q] - qM_i[p]]\, dx\, dt = \int_\sigma P_{1i}(x, t)\, dt - P_{2i}(x, t)\, dx$$

is valid in the region G where σ is a contour enclosing the region G, and

$$P_{1i}(x, t) = \frac{1}{2}\left[p\frac{\partial q}{\partial t} - q\frac{\partial p}{\partial t}\right] + b_i pq, \quad P_{2i}(x, t) = \frac{1}{2}\left[p\frac{\partial q}{\partial x} - q\frac{\partial p}{\partial x}\right] + c_i qp$$

Since the region G is a rectangle, we find

$$\iint_G pL_i[q]\,dxdt = \int_0^T [P_{1i}(l, t) - P_{1i}(0, t)]\,dt + \int_0^l [P_{2i}(x, T) - P_{2i}(x, 0)]\,dx + \iint_G qM_i[p]\,dx\,dt$$

Integrating by parts the one-dimensional integrals on the right side of this equation, we obtain

$$\iint_G pL_i[q]\,dxdt = [p(l, t)\,q(l, t)]_{t=0}^{T} - [p(0,t)\,q(0,t)]_{t=0}^{T} - \int_0^l \left\{q(x, t)\left[\frac{\partial p}{\partial x} - c_i p\right]\right\}_{t=0}^{T} dx - \int_0^T \left\{q(x, t)\left[\frac{\partial p}{\partial t} - b_i p\right]\right\}_{x=0}^{l} dt + \iint_G qM_i[p]\,dxdt \quad (11)$$

In the last equation we set $q = \Delta u_i$, $p = \Delta w_i$. Then, by virtue of equations (5) and conditions (6) and (8), it follows that

$$\iint_G \sum_{i=1}^{n} w_i L_i[\Delta u_i]\,dxdt = -\sum_{i=1}^{n}\left[A_i \Delta u_i(l, T) + \int_0^l \alpha_i(x)\,\Delta u_i(x, T)\,dx + \int_0^T \beta_i(t)\,\Delta u_i(l, t)\,dt + \iint_G \gamma_i(x, t)\,\Delta u_i(x, t)\,dxdt\right] + \iint_G \sum_{i=1}^{n} \frac{\partial H}{\partial u_i}\Delta u_i(x, t)\,dx\,dt$$

or

$$\iint_G \sum_{i=1}^{n} w_i L_i[\Delta u_i]\,dxdt = -\Delta S + \iint_G \sum_{i=1}^{n} \frac{\partial H}{\partial u_i}\Delta u_i(x, t)\,dx\,dt \quad (12)$$

where ΔS is the increment added to the functional S as a result of the change from the control function $v(x, t)$ to the control function $v + \Delta v$. Moreover, we have

$$\iint_G \sum_{i=1}^{n} \Delta w_i L_i[u_i]\,dx\,dt = \iint_G \sum_{i=1}^{n} \frac{\partial H}{\partial w_i}\Delta w_i\,dx\,dt \quad (13)$$

In equation (11) we set $q = \Delta u_i$, $p = \Delta w_i$. Then, by virtue of equation (7) and the boundary conditions (8), it follows that

$$\iint_G \sum_{i=1}^{n} \Delta w_i L_j[\Delta u_i]\,dx\,dt = \iint_G \sum_{i=1}^{n} \Delta\frac{\partial H}{\partial u_i}\Delta u_i\,dx\,dt$$

On the other hand, equation

$$\iint_G \sum_{i=1}^{n} \Delta w_i L_i \,[\Delta u_i]\, dx\,dt = \iint_G \sum_{i=1}^{n} \Delta \frac{\partial H}{\partial w_i} \Delta w_i \, dx\, dt$$

also holds.

From the last two formulas we find that

$$\iint_G \sum_{i=1}^{n} \Delta w_i L_i \,[\Delta u_i]\, dx\,dt = \frac{1}{2} \iint_G \sum_{i=1}^{2n} \Delta \frac{\partial H}{\partial z_i} \Delta z_i \, dx\, dt \tag{14}$$

where z denotes a $2n$-dimensional vector with components $u_1, \ldots, u_n$, $w_1, \ldots, w_n$ and $\Delta \partial H/\partial z_i$ is determined from formula (9). We apply Taylor's formula to the function H, neglecting all terms of order higher than the second in the increments

$$H(z + \Delta z, v + \Delta v, x, t) - H(z, v, x, t) = H(z, v + \Delta v, x, t) - H(z, v, x, t) + \\ + \sum_{i=1}^{2n} \frac{\partial H(z, v + \Delta v, x, t)}{\partial z_i} \Delta z_i + \frac{1}{2} \sum_{i,j=1}^{2n} \frac{\partial^2 H(z + \theta \Delta z, v + \Delta v, x, t)}{\partial z_i \partial z_j} \Delta z_i \Delta z_j \tag{15}$$

where $0 \leqslant \theta \leqslant 1$. From equation (10), by virtue of formulas (12) to (15), it follows that

$$\Delta J = -\Delta S + \iint_G \sum_{i=1}^{2n} \frac{\partial H(z, v, x, t)}{\partial z_i} \Delta z_i \, dx\, dt + \\ + \frac{1}{2} \iint_G \sum_{i=1}^{2n} \Delta \frac{\partial H(z, v, x, t)}{\partial z_i} \Delta z_i \, dx dt - \iint_G \sum_{i=1}^{2n} \frac{\partial H(z, v + \Delta v, x, t)}{\partial z_i} \Delta z_i \, dx\, dt - \\ - \frac{1}{2} \iint_G \sum_{i,j=1}^{2n} \frac{\partial^2 H(z + \theta \Delta z, v + \Delta v, x, t)}{\partial z_i \partial z_j} \Delta z_i \Delta z_j \, dx\, dt - \\ - \iint_G H(z, v + \Delta v, x, t) - H(z, v, x, t)]\, dx dt = 0$$

Collecting terms and applying Taylor's formula to the function $\partial H/\partial z_i$, we obtain

$$\Delta S = -\iint_G [H(z; v + \Delta v, x, t) - H(z, v, x, t)]\, dx dt + \eta \qquad (\eta = \eta_1 + \eta_2) \tag{16}$$

$$\eta_1 = \frac{1}{2} \iint_G \sum_{i=1}^{2n} \left[\frac{\partial H(z, v + \Delta v, x, t)}{\partial z_i} - \frac{\partial H(z, v, x, t)}{\partial z_i}\right] \Delta z_i \, dx\, dt$$

$$\eta_2 = \frac{1}{2}\iint\limits_G \sum_{i,j=1}^{2n} \left[\frac{\partial^2 H(z+\theta\,\Delta z,\ v+\Delta v,\ x,\ t)}{\partial z_i \partial z_j} - \frac{\partial^2 H(z+\theta_1\Delta z,\ v+\Delta v,\ x,\ t)}{\partial z_i \partial z_j}\right]\Delta z_i \Delta z_j\, dx\, dt \qquad (0 \leqslant \theta_1 \leqslant 1)$$

Formula (16) defines the increment of the functional S when the control function changes from $v(x, t)$ to $v + \Delta v$. In order to estimate the remainder η, we consider the first n equations of (7) and introduce the auxiliary variables ω_i, setting

$$\frac{\partial \Delta u_i}{\partial x} + c_i \Delta u_i = \omega_i$$

Then these equations may be represented in the form

$$\frac{\partial \omega_i}{\partial t} + b_i\omega_i = r_i \Delta u_i + \Delta\frac{\partial H}{\partial w_i} \qquad \left(r_i = b_i c_i + \frac{\partial c_i}{\partial t}\right)$$

From this, by virtue of conditions (8), it follows that

$$\Delta u_i(x, t) = \int_0^x \omega_i(\xi, t) \exp\left(\int_x^\xi c_i(\xi, t)\, d\xi\right) d\xi \tag{17}$$

$$\omega_i(x, t) = \int_0^t \left[r_i(x, \tau)\ \Delta u_i(x, \tau) + \Delta\frac{\partial H}{\partial w_i}\right] \exp\left(\int_t^\tau b_i(x, \tau)\, d\tau\right) d\tau \tag{18}$$

Since the functions b_i and r_i are bounded in the region G, and the functions $\partial H/\partial w_i$ satisfy a Lipschitz condition with respect to the arguments u_k and v_j, we find from (18)

$$|\omega_i(x, t)| \leqslant \int_0^t \left[N_1 \sum_{j=1}^n |\Delta u_j(x, \tau)| + N_2 \sum_{k=1}^r |\Delta v_k(x, \tau)|\right] d\tau \qquad (i = 1, \ldots, n)$$

where N_1 and N_2 are specified positive constants. Using this estimate, from equation (17) we find

$$|\Delta u_i(x, t)| \leqslant P \int_0^l \int_0^t \left[N_1 \sum_{j=1}^n |\Delta u_i(x, \tau)| + N_2 \sum_{k=1}^r |\Delta v_k(x, \tau)|\right] d\tau\, dx \tag{19}$$

where P is a specified positive constant. Integrating this inequality with respect to x from 0 to l and summing over all i, we find

$$U(t) \leqslant A + B\int_0^t U(\tau)\, d\tau \qquad (B = nlPN_1)$$

$$U(t) = \sum_{i=1}^n \int_0^l |\Delta u_i(x, t)|\, dt, \quad A = nlN_2 \iint\limits_G \sum_{k=1}^r |\Delta v_k(x, t)|\, dx\, dt$$

We make use of Lemma 1 of [11, p. 19], according to which the function $U(t)$ will satisfy the inequality

$$U(t) \leqslant Ae^{Bt}$$

From this it follows that

$$\sum_{j=1} \int_0^l |\Delta u_j(x, t)| \, dxdt \leqslant nlPN_2 e^{BT} \iint_G \sum_{k=1}^r |\Delta v_k(x, t)| \, dx\, dt$$

Consequently, from the inequalities (19) we find

$$|\Delta u_i(x, t)| \leqslant R_1 \iint_G \sum_{k=1}^r |\Delta v_k(x, t)| \, dx\, dt$$

where R_1 is a specified positive constant.

A similar estimate can be found for $|\Delta w_i(x, t)|$. Consequently

$$|\Delta z_i(x, t)| \leqslant R \iint_G \sum_{k=1}^r |\Delta v_k(x, t)| \, dx\, dt \qquad (i = 1, \ldots, 2n) \tag{20}$$

where R is a positive number such that $R_1 \leqslant R$.

Formula (16) for the increment of the functional S and the inequalities (20) are similar to the corresponding formulas in [5]. Consequently, the validity of the maximum principle we have formulated is readily proved by repeating almost literally the proof of Theorem 1 of that article.

Although the theorem we have proved does not provide any sufficient conditions for the existence of optimal control functions, it enables us to find, among all the solutions of the boundary value problem (1) to (2), individual isolated solutions which satisfy the maximum conditions. In fact, to solve the problem by the maximum principle, we must determine $2n + 1$ unknowns u_i, w_i and v from the $2n + 1$ equations (1), (3) and (H). Consequently, we have a "complete" system of equations for finding u_i, w_i and v. The first $2n$ equations are second-order differential equations. In solving them, therefore, we shall encounter arbitrary functions which can be eliminated by means of the supplementary conditions (2) and (6). We thereby determine the set of isolated solutions of the boundary value problem (1) to (2) which satisfy the conditions of the maximum principle. If we find that there are a finite number of such solutions and it is clear from physical conditions of the problem that optimum control functions exist, then we should expect some of these solutions to be optimal.

In the case where the system of equations (1) is linear

$$L_i\,[u_i] = \sum_{k=1}^{n} a_{ik}\,(x,\,t)\,u_k + \varphi_i\,(v) \qquad (i = 1, \ldots, n) \tag{21}$$

the following theorem holds.

Theorem 2. In order that the control function $v(x,\ t)$ in the system of equations (21) be min-optimal (max-optimal) with respect to S in the local sense, it is necessary and sufficient that it satisfy a maximum (minimum) condition.

The proof of this theorem is almost literally the same as the proof of the corresponding theorem in [5].

Example 1. Let a controlled process be described by the equation

$$\frac{\partial^2 u}{\partial t\,\partial x} + 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = -\,2u + v, \qquad 0 < x \leqslant 2, \qquad 0 < t \leqslant T \tag{22}$$

where v i he controlling parameter, with $|v| \leqslant 1$. It is required to find a contiol function $v(x,\ t)$, $0 \leqslant x \leqslant 2$, $0 \leqslant t \leqslant T$, such that for the solution of equation (22) that corresponds to this control function and satisfies the conditions

$$u\,(x,\,0) = u\,(0,\,t) = 0 \tag{23}$$

the functional

$$S = \int\limits_0\int\limits_0 (x - 1)\,u\,(x,\,t)\,dx\,dt$$

should reach a minimum.

We define the function $w(x,\ t)$ by means of the equation

$$\frac{\partial^2 w}{\partial x\,\partial t} - 2\frac{\partial w}{\partial x} - \frac{\partial w}{\partial t} = -\,2w - (x - 1)$$

and the supplementary conditions $w(x,\ T) = w(2,\ t) = 0$. This function will be of the form

$$w\,(x,\,t) = \frac{1}{2}\,(2e^{x-2} - x)\,(1 - e^{2(t-T)})$$

We construct the function

$$H = w\,(-\,2u + v)$$

By the maximum principle, the optimal control function will be determined in accordance with the formula

$$v\,(x,\,t) = \operatorname{sign} \frac{1}{2}\,(2e^{x-2} - x)\,(1 - e^{2(t-T)}) \qquad (0 \leqslant x \leqslant 2,\, 0 \leqslant t \leqslant T)$$

or

$$v\,(x,\,t) = \operatorname{sign}\,(2e^{x-2} - x)$$

Here the expression in parentheses is equal to zero at only one point y of the interval $(0,\ 2)$, where $y < 1$; we have $2e^{x-2} - x > 0$ if $x < y$ and $2e^{x-2} - x < 0$ if $x > y$. Therefore

$$v\,(x,\,t) = \begin{cases} 1 & (0 \leqslant x < y,\quad 0 \leqslant t \leqslant T) \\ -1 & (y < x \leqslant 2,\quad 0 \leqslant t \leqslant T) \end{cases} \tag{24}$$

and consequently the solution of the problem (22) for $v = v(x,\ t)$ is of the form

$$u\,(x,\,t) = \begin{cases} {}^1/_2\,(1 - e^{-x})\,(1 - e^{-2t}) & \text{if } 0 \leqslant x \leqslant y,\ 0 \leqslant t \leqslant T \\ {}^1/_2\,(1 - e^{-2t})\,(2e^{-(x-y)} - e^{-x} - 1) & \text{if } y \leqslant x \leqslant 2,\ 0 \leqslant t \leqslant T \end{cases}$$

On this solution

$$S = {}^1/_4\,[2T + e^{-2T} - 1]\,[y^2 - 4e^{y-2} + 2e^{-2}] \tag{25}$$

or, if we take into account the fact that

$$y = 2e^{y-2} \tag{26}$$

we find

$$S = {}^1/_4\,[2T + e^{-2T} - 1]\,[y^2 - 2y + 2e^{-2}]$$

A direct check will convince us that if in formula (25) the quantity S is regarded as a function of the variable y, varying over the interval $[0,\ 2]$, then S will reach its minimum at the point y at which equation (26) is satisfied. This means that, among all the control functions of the form (24), the control function $v(x,\ t)$ for which condition (26) is satisfied at the point y will give the functional its minimum value. If we apply Theorem 2, we find that this control function is min-optimal.

Example 2. Let a controlled process be described by the equation

$$\frac{\partial^2 u}{\partial x\,\partial t} = v, \quad |\,v\,| \leqslant 1, \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant t \leqslant 1 \tag{27}$$

with the supplementary conditions

$$u\,(x,\,0) = u\,(0,\,t) = 0 \tag{28}$$

It is required to determine the admissible control function for which the functional

$$S = \int_0^1 u\,(x,\,1)\,dx - \int_0^1 u\,(1,\,t)\,dt$$

reaches its minimum value, and to find that minimum value of S.

To find the function $w(x,\ t)$ we can make use of the equation $\partial^2 w/\partial x \partial t = 0$ and the supplementary conditions $w(x,\ 1) = x - 1$, $w(1,\ t) = 1 - t$.

Consequently, $w(x,\ t) = x - t$, and from the maximum condition we find that

$$v\,(x,\,t) = -1\ (t > x), \qquad v\,(x,\,t) = 1\ (t < x)$$

Thus, we obtain two equations for finding the function $u(x,\ t)$

$$\frac{\partial^2 u}{\partial x\,\partial t} = 1 \quad (t - x < 0), \qquad \frac{\partial^2 u}{\partial x\,\partial t} = -1 \quad (t - x > 0) \tag{29}$$

with the supplementary conditions (28). From this we find that

$$u\,(x,\,t) = \begin{cases} -\,xt + \varphi\,(x) & (0 \leqslant x \leqslant t \leqslant 1) \\ \quad xt - 2t^2 + \varphi\,(t) & (0 \leqslant t \leqslant x \leqslant 1) \end{cases}$$

where $\varphi(x)$ is an arbitrary differentiable function, with $\varphi(0) = 0$. The corresponding value of the functional S is $-1/3$.

Thus, in the example considered the solution corresponding to the optimum control function is a function $u(x,\ t)$ that depends on an arbitrary function φ, while the value of the functional S is independent of φ. Therefore, keeping the value of S fixed on $u(x,\ t)$, we can impose an additional condition. For example, we may require the function $u(x,\ t)$ to have continuous derivatives $\partial u/\partial x$ and $\partial u/\partial t$ on the switching line $x = t$.

Then

$$u\,(x,\,t) = \begin{cases} x\,(x - t) & (0 \leqslant x \leqslant t \leqslant 1) \\ t\,(x - t) & (0 \leqslant t \leqslant x \leqslant 1) \end{cases}$$

In the example considered the functional S is independent of the arbitrary function φ appearing in the solution $u(x,\ t)$, which corresponds to the regular optimal control function $v(x,\ t)$. In the general case it may depend on arbitrary functions. Therefore, if we wish to apply the maximum principle proved above, we should assing to regular control functions only continuously differentiable solutions of the

Goursat problem with predetermined conditions on the continuity of the derivatives. The above method of investigation may be applied to the study of optimal processes that can be described by hyperbolic equations with supplementary initial conditions and some forms of boundary conditions.

Example 3. Let a controlled process be described by the equation

$$L[u] \equiv \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x, t, u, v)$$

where a^2 is a positive constant, v is a controlling parameter, and f is continuous in x and t and is twice continuously differentiable with respect to u and v. Let the function u, determined by this equation, satisfy the supplementary conditions

$$u(x, 0) = \varphi_0(x), \qquad u_t'(x, 0) = \varphi_1(x), \qquad u(0, t) = \psi_0(t), \qquad u_x'(l, t) = \psi_1(t)$$

where $\varphi_i(x)$ and $\psi_i(t)$ are continuously differentiable functions satisfying the matching conditions. The class of optimal control functions is determined in the same manner as in the problem considered above. We define the functional S by the formula

$$S = \int_0^l \alpha(x)\, u(x, T)\, dx + \int_0^T \beta(t)\, u(l, t)\, dt + \iint_G \gamma(x, t)\, u(x, t)\, dx\, dt$$

In this case the role of the system (3) is played by the equation

$$L[w] = w\, \partial f / \partial u - \gamma(x, t)$$

and the supplementary conditions for the function $w(x, t)$ must be taken in the form

$$w(x, T) = 0, \qquad \partial w(x, T) / \partial t = \alpha(x), \qquad w(0, t) = 0, \qquad w(l, t) = \beta(t)$$

The function H is of the form

$$H(w, u, v, x, t) = wf(x, t, u, v)$$

By the same reasoning as we applied to the study of the Goursat problem, we can readily establish the validity of Theorems 1 and 2.

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